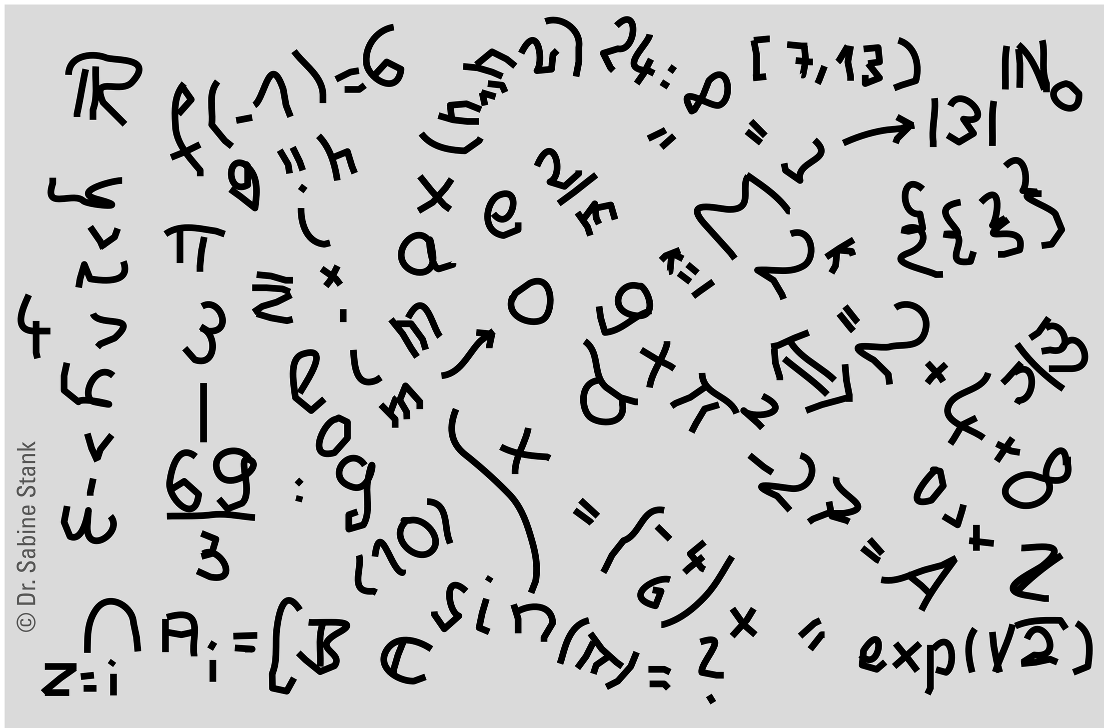
Ostfalia
Hochschule für angewandte Wissenschaften

Fakultät Versorgungstechnik

An Ant, Russian Dolls, Even a Cow Can Help:

Understanding Mathematics by Visualizing Structures

A novice see mathematics ...



... the expert recognizes structures.

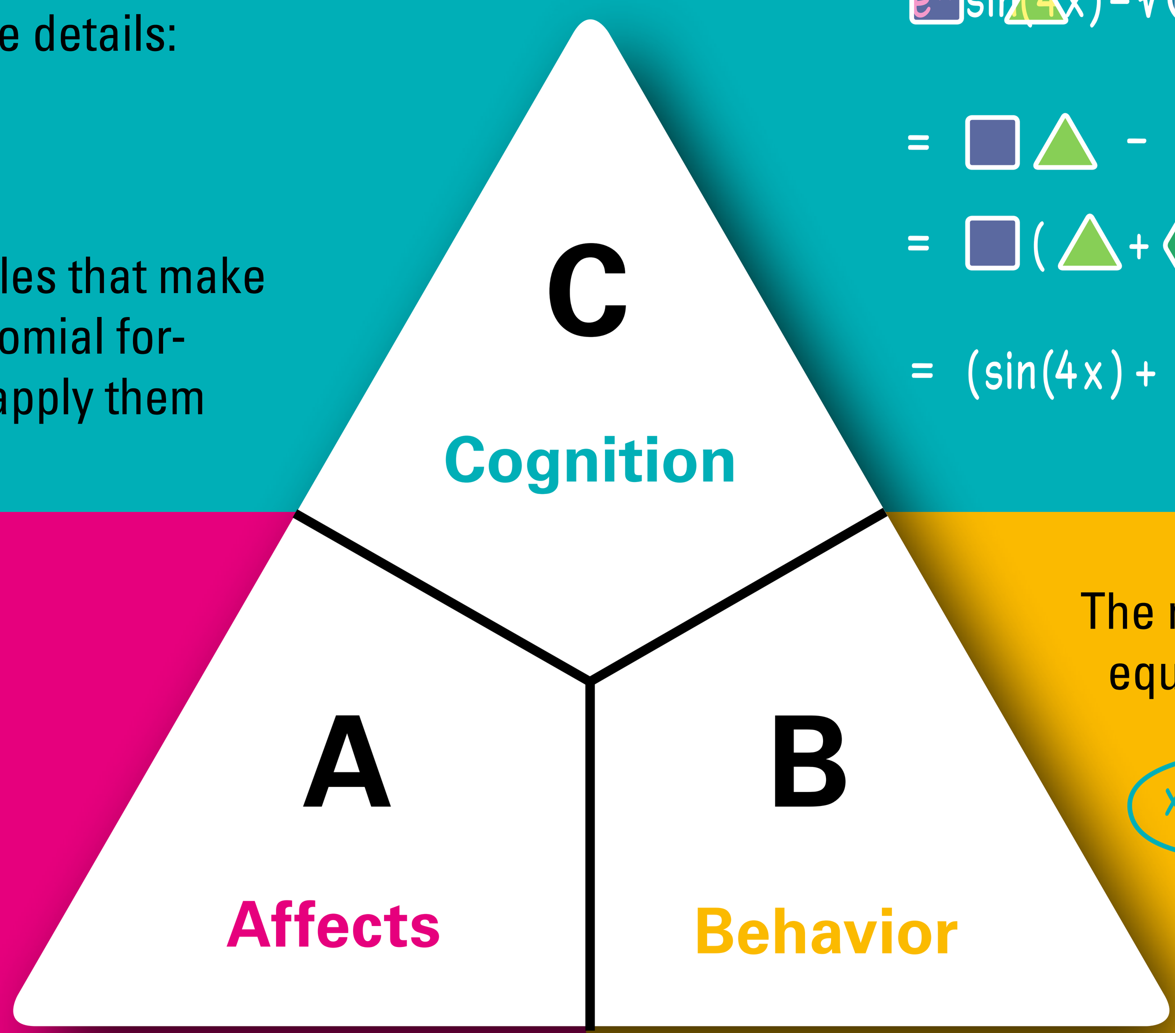
To understand and solve mathematical problems involves, among other things, recognizing the underlying abstract structures and patterns. Expert have stored these in the form of internal images and they are available as comparison models for new structures. „Abstract“ means that these structures are not obvious, but must be „seen“ by comparison with the inner images. Novices must therefore first acquire structures in the form of inner images, whereby the necessary learning process can be supported by the expert.

Visualization of structures in a relationship that takes into account Affect, Behavior and Cognition (ABC of psychology)

AFFECTS: Visualization of structures by mapping abstract elements to real objects or persons and embedding them into a story

BEHAVIOR: Visualization of action structures by surprising analogies

COGNITION: Visualization of structures with symbols

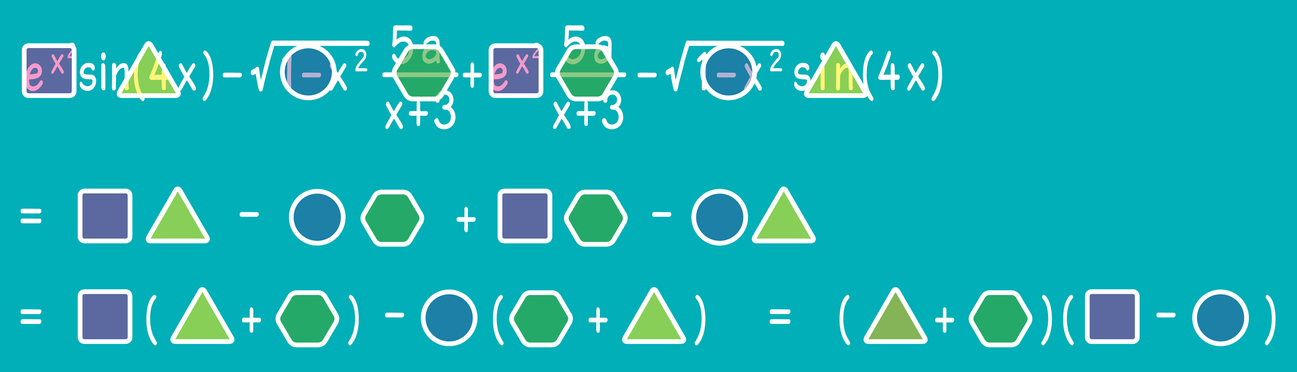


The novices are to write the following term as a product:

$$e^{x^2}\sin(4x)-\sqrt{1-x^2}\,\frac{5a}{x+3}+e^{x^2}\frac{5a}{x+3}-\sqrt{1-x^2}\,\sin(4x)$$

OBSERVATION:
Novices often do not know how to start, they only see details: e, x, 2, -, sin, 4, (etc.

COMMENT:
A prerequisite for the solution is the knowledge of rules that make products from sums: The distributive law and the binomial formulas. These are known to novices, but they do not apply them because they do not see the underlying structure of the term. It is helpful to use symbols that work like containers, which can be filled with arbitrary terms :



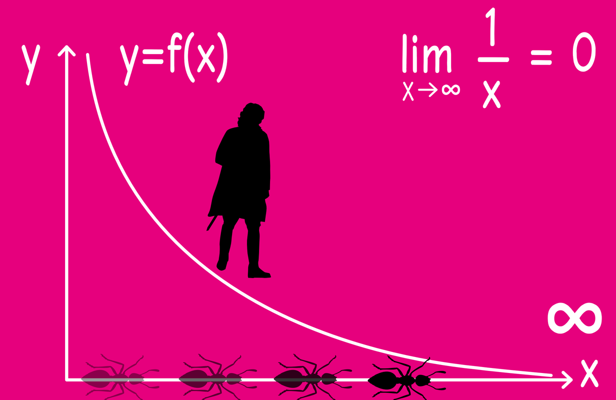
$$= \left(\sin(4x) + \frac{5a}{x+3}\right)\left(e^{x^2}-\sqrt{1-x^2}\right)$$ is the result.

The novices are to determine the following limit:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1}{x} = ?$$

OBSERVATION:
The structures in this notation are not read out and related (assignment, covariation), so that the limit value can not be determined.

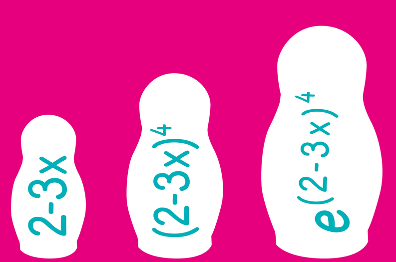
COMMENT:
Here an ant and Mr. Graph can help: The ant crawls on the x-axis and Mr. Graph moves on the graph of $f(x)=\frac{1}{x}$. The farther the ant creeps to the right, the closer is Mr. Graph. Once the ant has arrived at infinity, both have their date.

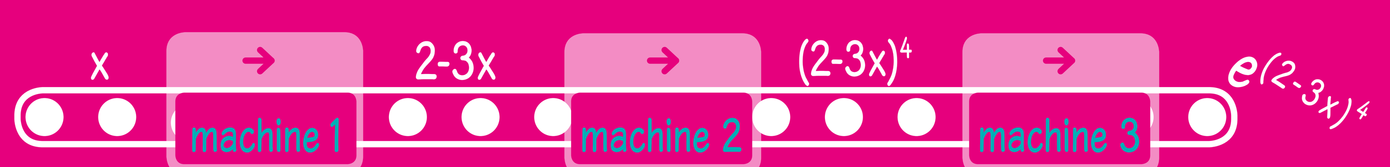


The novices are to determine the first derivation of the function $f(x) = e^{(2-3x)^4}$

OBSERVATION:
They can´t use the chain rule since they do not capture the structure.

COMMENT:
Russian dolls help to recognize the nested structure of the function: Another visualization of the inner, middle and outer function uses the succession of several machines:





The novices have solved the following equations with the pq-formula:

1) $x^2-8x+13=0$
2) $-\frac{1}{2}x^2-x+12=0$
3) $(x-5)x-3=2$

$$x^2+px+q=0 \quad x_{1/2}=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$$

Next, the following equation is to be solved: $x^2-5x=0$

OBSERVATION:
The novices calculate: $x_{1/2}=\frac{5}{2}\pm\sqrt{\left(\frac{5}{2}\right)^2-0}=\frac{5}{2}\pm\frac{5}{2}$ $\quad x_1=5 \quad x_2=0$

The novices don't see the possibility of applying the distributive law and the null product rule: $x^2-5x=0$, $x(x-5)=0$ $\quad x_1=5 \quad x_2=0$

COMMENT:
Here we see how an initial stimulus (application of the pq-formula) activates an association field, with which the following is then associated.

In order to make this unconscious structure of action aware of the acting person, the following question-response game has proven itself:

What color has snow? — White!
What is the color of the clouds? — White!
What color is the paper? — White!
What does the cow drink? — Milk!

This is, of course, the wrong answer! The cow drinks water!

Here, the cow helps to make structures of action visible and thus make them changeable!

by **Dr. Sabine Stank**